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# A Generalisation of Planar Magnetic Gradiometer Design via Orthogonal Polynomials

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We describe a problem in magnetic field detection involving a form of spatial filtering to detect weak signal sources in the presence of noise. Conventionally $N$-th order magnetic field gradiometers of fixed geometry are used in this situation. The pre-defined geometry completely determines the spatial sensitivity of such gradiometers. We demonstrate a method of making such devices much more flexible in that the near-source response can be modified while maintaining gradiometric order. The problem is described by the solution of $N$ equations in sums and differences of powers, up to order $N$, of $m$ variables, with $m \geq N$. The values of $(m - N)$ variables are chosen on physical considerations. We show that when values of the $m$ variables are a solution set, they may be expressed as the roots of two polynomial equations, whose order is no greater than $(m + 1)/2$ when $m$ is odd, or $m/2$ when $m$ is even. These polynomial equations can be expressed as a linear combination of Chebyshev polynomials of the first and second kinds in the case of $m$ odd, and a related pair, fully described, in the case of $m$ even. Existence of, and bounds on, solution sets are discussed and examples given. © 1995 Academic Press, Inc.

## 1. INTRODUCTION

Magnetic field gradiometers coupled to Superconducting Quantum Interference Devices (SQUIDs) are widely used for the detection of small fields from local sources in the presence of much larger interference fields from more distant sources. This situation arises, for example, in non-destructive testing (NDT) [1, 2] and in biomagnetism, the study of magnetic fields from biological sources such as electrical activity in the brain [3, 4]. Typical field magnitudes are 0.2 pT for stimulus-evoked brain activity. Ambient noise levels vary widely depending on location and whether magnetic shielding is used, but white noise of 1 nT $Hz^{-1/2}$ with discrete frequency interference of up to 50 $\mu$T is typical. A bandwidth of at least 100 Hz is required for most medically useful measurements.

The field from magnetic sources generally decays with distance $r$ from the source as $r^{-n}$, where $1 < n < 3$ depends on the precise nature of the source. Thus typically signal sources in NDT are magnetic dipoles ($n = 3$) and those in biomagnetism are current dipoles ($n = 2$). The sources are normally external to the sensor structure. Discrimination against distant sources can be achieved by a device geometry responsive primarily to the gradient or higher spatial derivatives of the field. In practice this is approximated by a form of spatial filtering. An $N$-th order gradiometer is an arrangement of pickup loops designed so that the first $N$ terms of a Taylor expansion of the field threading the gradiometer couple zero net magnetic flux into it.

Early gradiometers, as illustrated in Fig. 1a, were fabricated by wire wound on a machined transformer, with complex mechanisms to compensate for geometrical imperfections. They were designed to sense field components in the $x$ (or radial) direction for two reasons. Firstly this component of the field has a simpler intuitive interpretation, and secondly, early simplistic models of biomagnetic sources indicated no $y$ or $z$ (tangential) components. In fact more realistic source models [5] and measurements using tangential field sensors [6] show the tangential and radial components to be comparable in magnitude. In the case of NDT all field components are of value.

The gradiometers discussed here sense tangential $z$ components of the field and its differentials along the $x$-axis (see Fig. 1b and Fig. 2). They are fabricated with precise geometrical definition using photolithographic thin-film patterning in a plane [3, 4], obviating the need for mechanical trimming. A simple gradiometer of this type is illustrated in Fig. 2. It consists of a series of adjacent rectangular pickup loops, made of superconducting thin-films, wired in series with an insulated crossover between each pair of loops. The dimensions are such that a desired signal source will normally lie in the interval $1 < x < 2$ (i.e., the source distance from the gradiometer should be comparable to the gradiometer size), and the source sensitivity is greatest in the plane of the gradiometer. The positions



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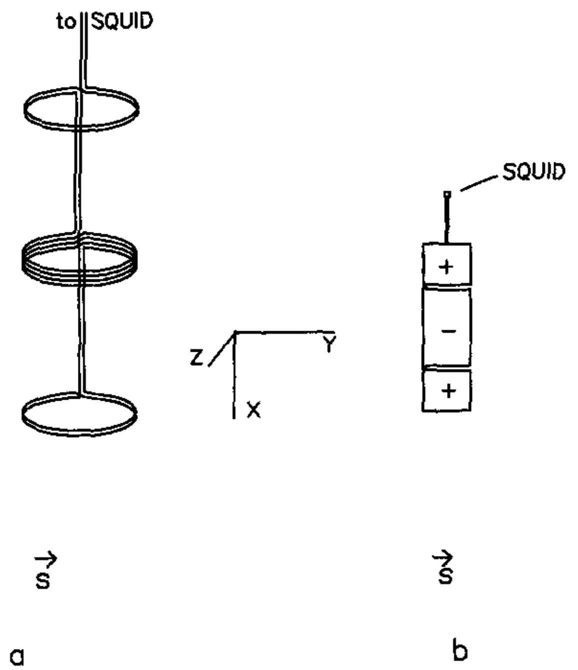


**FIG. 1.** Gradiometer geometries of (a) conventional wirewound axial field sensor and (b) planar tangential field sensor. Note the alternating winding directions. Typical source positions are marked *S*.

and number of the cross-overs determine not only the gradiometer order, but also the detailed response to nearby sources [7, 8]. In particular, the number of cross-overs, $m$, where $m \geq N$ [10], should be minimised, subject to other considerations, to minimise the gradiometer self-inductance. This, due to details of SQUID operation, maximises sensitivity [5].

This paper addresses the problem of selection of cross-over positions to achieve a given gradiometric order $N$ in the case where one or more cross-over positions have been pre-selected to achieve a given near-source response. This is required to give two specific properties. Firstly, it is highly desirable to locate the SQUID near the gradiometer on the same substrate, typically in the interval $-2 < x < -1$ (Fig. 2). However, the SQUID itself, as a superconductor, constitutes a nearby magnetic anomaly; i.e., it causes local distortion of the quasi-uniform interference fields and thus behaves as a local noise source. If the position of the cross-over nearest the SQUID (i.e., $x_2$ in Fig. 2) can be adjusted, a null-response to this source can be designed in. Secondly, the position of the cross-over nearest the signal source ($x_1$ in Fig. 2) can be adjusted to give a peak in the gradiometer sensitivity as a function of distance at $x > 1$. This can be used either for depth-profiling (by moving the gradiometer along the $x$-axis) or for discrimination against an overlying source (e.g., detection of a nerve underlying surface muscle). The method is however quite general and may be applied for any order $N$ and number of crossovers $m$. It reduces the problem from a computationally intensive $N$-dimensional iterative process to the trivial one of computing the roots of two real polynomials.

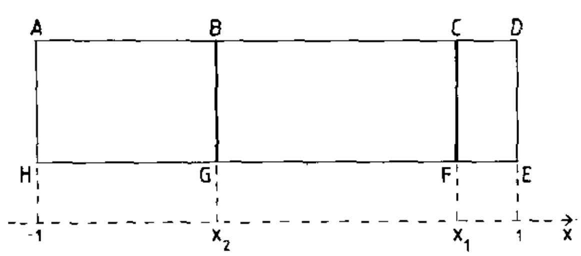


**FIG. 2.** Example of a planar gradiometer geometry. Insulated crossovers exist at $x_1$ and $x_2$; thus the only continuous current-carrying path is ABGFCDEFCBGHA. The structure is fabricated with wires or thin-film metal tracks, and any electric current induced by magnetic flux is monitored.

Denoting the cross-over positions as $x_s$, $s = 1,2, \ldots, m$, we require that

$$\begin{aligned} x_1 - x_2 + x_3 - \cdots - (-1)^m x_m &= \frac{1}{2}(1 + (-1)^m) \\ x_1^2 - x_2^2 + x_3^2 - \cdots - (-1)^m x_m^2 &= \frac{1}{2}(1 + (-1)^{m+1}) \qquad (1) \\ &\vdots \\ x_1^N - x_2^N + x_3^N - \cdots - (-1)^m x_m^N &= \frac{1}{2}(1 + (-1)^{m+N-1}), \end{aligned}$$

where $N \leq m$, and that the $x_s$'s also satisfy the inequalities

$$-1 \leq x_m < x_{m-1} < \cdots < x_2 < x_1 \leq 1. \qquad (2)$$

(See [2, 3, 4].)

We define a *solution set*, $\{x_s\}$, to be a set of values which satisfy both Eqs. (1) and the inequalities (2).

When $N = m$, it can be shown [10, 12] that Eqs. (1) may be solved to give the single solution set

$$\left\{ x_s = \cos\left(\frac{s\pi}{m+1}\right) \right\}.$$

When $N$ is less than $m$ there is, of course, no unique solution set; for any solution set, $N$ of the $x_s$'s will depend on $m - N$ of the $x_s$'s. Values $x_1, x_2, \ldots, x_m$ satisfying Eqs. (1) may be found by assigning values to $m - N$ of the $x_s$'s and solving (1) to obtain the remaining $x_s$'s—although algebraic solution of Eqs. (1) proves intractable for large $m$. If the set of values so obtained also satisfies (2), it is a solution set.

For $x_s$'s satisfying (1), the obvious interchangeability of $x_{2i-1}$ with $x_{2j-1}$, and $x_{2i}$ with $x_{2j}$, leads us to examine two polynomial equations with roots $\{x_{2i-1}\}$ and $\{x_{2i}\}$. We give these polynomial equations, which take different forms for $m = 2n - 1$ and $m = 2n$ ($n \in \mathbb{N}$): these cases are treated separately. As expected, the coefficients of these equations depend upon the values of

$m - N$ constants. Thus, we reduce the problem of finding solutions for Eqs. (1) to that of obtaining the roots of two polynomial equations.

We illustrate the use of this formalism by examining the behaviour of the $x_s$'s for $N = m - 1$, when $m - 1$ of these depend on the value of one $x_s$; we show when it is possible to solve Eqs. (1) to give a valid solution set, and obtain bounds for each $x_s$.

## 2. SOLUTIONS WHEN $m = 2n - 1$

For a solution set $\{x_s\}$, we consider the polynomial equations

$$x^n + p_1 x^{n-1} + \cdots + p_{n-1} x + p_n = 0$$

and

$$x^{n-1} + g_1 x^{n-2} + \cdots + g_{n-2} x + g_{n-1} = 0$$

which have roots $x_1, x_3, \ldots, x_{2n-1}$ and $x_2, x_4, \ldots, x_{2n-2}$ respectively. We write these equations in the form

$$T_n(x) + k_1 T_{n-1}(x) + \cdots + k_{n-1} T_1(x) + k_n = 0 \qquad (3)$$

and

$$U_{n-1}(x) + m_1 U_{n-2}(x) + \cdots + m_{n-2} U_1(x) + m_{n-1} = 0, \qquad (4)$$

where $T_1, T_2, \ldots, T_n$ are Chebyshev polynomial of the first kind, and $U_1, U_2, \ldots, U_{n-1}$ are Chebyshev polynomials of the second kind (see [9]). As the roots of these equations are to be a solution set, the constants $k_i$ and $m_j$ are real and are such that the $n$ roots of (3) and the $n - 1$ roots of (4) all lie in the interval $[-1, 1]$.

We now show that

$$k_i = m_i \text{ for all } i = 1, 2, \ldots, N$$

when

$$N \in \{1, 2, \ldots, n - 1\}$$

and

$$k_i = m_i \qquad \text{for } i = 1, 2, \ldots, n - 1$$

with

$$k_i = 0 = m_i \quad \text{for } i = 2n - N, 2n - N + 1, \ldots, n$$

when

$$N \in \{n, n + 1, \ldots, 2n - 1\}$$

so that (3) and (4) are two polynomial equations with $m - N$ independent coefficients.

As $x_i \in [-1, 1]$, we begin by defining new variables

$$x_{2i-1} = \cos \theta_i \quad (i = 1, 2, \ldots, n)$$

and

$$x_{2i} = \cos \alpha_i \quad (i = 1, 2, \ldots, n - 1).$$

By considering

$$\begin{aligned} W_r &= \sum_{i=1}^{n} \cos r\theta_i - \sum_{i=1}^{n-1} \cos r\alpha_i \\ &= \sum_{i=1}^{n} T_r(\cos \theta_i) - \sum_{i=1}^{n-1} T_r(\cos \alpha_i) \\ &= \frac{r}{2} \sum_{j=0}^{[r/2]} (-1)^j \frac{(r-1-j)!}{j!(r-2j)!} 2^{r-2j} \left[ \sum_{i=1}^{n} x_{2i-i}^{r-2j} - \sum_{i=1}^{n-1} x_{2i}^{r-2j} \right] \end{aligned}$$

(see [9]) for $r = 1, 2, \ldots, N$, it can be seen from Eqs. (1) that

$$W_r = \begin{cases} T_r(0) & \text{when } r = 1, 3, 5, \ldots \\ T_r(1) & \text{when } r = 2, 4, 6, \ldots \end{cases}$$

and so the $\theta_i$ and $\alpha_i$ must satisfy the equations

$$\begin{aligned} &\cos \theta_1 + \cos \theta_2 + \cdots + \cos \theta_n \\ &\quad = \cos \alpha_1 + \cos \alpha_2 + \cdots + \cos \alpha_{n-1} \\ &\cos 2\theta_1 + \cos 2\theta_2 + \cdots + \cos 2\theta_n \\ &\quad = \cos 2\alpha_i + \cos 2\alpha_2 + \cdots + \cos 2\alpha_{n-1} + 1 \\ &\quad \vdots \\ &\cos N\theta_1 + \cos N\theta_2 + \cdots + \cos N\theta_n \\ &\quad = \cos N\alpha_1 + \cos N\alpha_2 + \cdots + \cos N\alpha_{n-1} \\ &\quad + \frac{1}{2}[1 + (-1)^N]. \end{aligned} \qquad (5)$$

With the transformation $X = \cos \theta$, Eq. (3) may be written (see [9]) as

$$\cos n\theta + k_1 \cos(n-1)\theta + \cdots + k_{n-1} \cos \theta + k_n = 0 \qquad (6)$$

which has roots $\cos \theta_1, \cos \theta_2, \ldots, \cos \theta_n$. With $X = \cos \alpha$, Eq. (4) may be written as

$$\frac{\sin n\alpha}{\sin \alpha} + m_1 \frac{\sin(n-1)\alpha}{\sin \alpha} + \cdots + m_{n-2}\frac{\sin 2\alpha}{\sin \alpha} + m_{n-1} = 0 \quad (7)$$

which has roots $\cos \alpha_1, \cos \alpha_2, \ldots, \cos \alpha_{n-1}$.

Consider the complex variable

$$z = \cos \theta + i \sin \theta$$

so that

$$z^r + z^{-r} = 2 \cos r\theta \quad \forall\, r \in \mathbb{N}.$$

Using this substitution, (6) may be written as

$$z^n + \frac{1}{z^n} + k_1\left[z^{n-1} + \frac{1}{z^{n-1}}\right] + \cdots + k_{n-1}\left[z + \frac{1}{z}\right] + 2k_n = 0$$

or

$$z^{2n} + k_1 z^{2n-1} + \cdots + k_{n-1} z^{n+1} + 2k_n z^n + k_{n-1} z^{n-1} + \cdots + k_1 z + 1 = 0. \quad (8)$$

The $2n$ complex roots of this equation occur as the complex conjugate pairs

$$z_{2i-1} = \cos \theta_i + i \sin \theta_i; \qquad z_{2i} = \overline{z_{2i-1}} = \cos \theta_i - i \sin \theta_i \quad (i = 1, 2, \ldots, n), \quad (9)$$

where the $\cos \theta_i$ are self-evidently the $n$ roots of (6). Moreover, it can be seen from (9) that

$$z_{2i-1}^r + z_{2i}^r = 2 \cos r\theta_i \quad \forall\, r \in \mathbb{N}$$

so that the $\cos \theta_1$ roots of (6) and the $z_i$ roots of (8) are such that

$$2\sum_{i=1}^{n} \cos r\theta_i = \sum_{i=1}^{2n} z_i^r \quad \forall\, r \in \mathbb{N}. \quad (10)$$

Now consider equation (7). With

$$z = \cos \alpha + i \sin \alpha$$

so that

$$z^r - z^{-r} = 2i \sin r\alpha \quad \forall\, r \in \mathbb{N}$$

Eq. (7) may be written as

$$\frac{1}{z^{n-1}}\left[\frac{z^{2n}-1}{z^2-1} + m_1 \frac{z(z^{2n-2}-1)}{z^2-1} + \cdots + m_{n-2}\frac{z^{n-2}(z^4-1)}{z^2-1} + m_{n-1} z^{n-1}\right] = 0. \quad (11)$$

It can be seen that the equation

$$z^{2n} + m_1 z^{2n-1} + \cdots + m_{n-1} z^{n+1} - m_{n-1} z^{n-1} - \cdots - m_1 z - 1 = 0 \quad (12)$$

has the same roots as equation (11), together with two additional roots at $z^2 = 1$. The complex roots of (12) again occur as complex conjugate pairs, and the $2n$ roots of (12) are given by

$$z_{2i-1} = \cos \alpha_i + i \sin \alpha_i \quad z_{2i} = \cos \alpha_i - i \sin \alpha_i \quad (i = 1, 2, \ldots, n-1)$$

$$z_{2n-1} = -1 \qquad z_{2n} = 1,$$

where again the $\cos \alpha_i$ are the $(n-1)$ roots of equation (7). Hence the $\cos \alpha_i$ roots of (7) and the $z_i$ roots of (12) are such that

$$2\sum_{i=1}^{n-1} \cos r\alpha_i = \sum_{i=1}^{2n-2} z_i^r = \sum_{i=1}^{2n} z_i^r \quad \text{when } r = 1, 3, 5, \ldots$$

and

$$2\sum_{i=1}^{n-1} \cos r\alpha_i = \sum_{i=1}^{2n-2} z_i^r = \left[\sum_{i=1}^{2n} z_i^r\right] - 2 \quad \text{when } r = 2, 4, 6, \ldots. \quad (13)$$

It follows from (5), (10) and (13) that the sums of the $r$th powers of the roots of (8) and (12) must be equal for $r = 1, 2, \ldots, N$.

Newton's Formula [11] state that for a polynomial equation

$$z^{2n} + p_1 z^{2n-1} + \cdots + p_{2n-1} z + p_{2n} = 0$$

the sums of the $r$th powers of the roots

$$S_r = \sum_{i=1}^{2n} z_i^r$$

are related to the coefficients $p_1, p_2, \ldots, p_{2n}$ by the relationships

$$
\begin{aligned}
&s_1 + p_1 = 0\\
&s_2 + p_1 s_1 + 2p_2 = 0\\
&s_3 + p_1 s_2 + p_2 s_3 + 3p_3 = 0\\
&\vdots\\
&s_{2n} + p_1 s_{2n-1} + \cdots + p_{2n-1} s_1 + 2n\, p_{2n} = 0
\end{aligned}
$$

and so for two such polynomial equations, the sums of the $r$th powers of the roots will be equal for all $r = 1, 2, \ldots, N$, ($N \leq 2n$) if and only if the first $N$ coefficients of the polynomials are equal.

Hence, as the $\cos\theta_i$ roots of (6) and the $\cos\alpha_i$ roots of (7) satisfy (5) it follows that the first N coefficients of (8) and (12) are equal, which implies

$$k_i = m_i \qquad \text{for } i = 1, 2, \ldots, N$$

when

$$N \in \{1, 2, \ldots, n - 1\}$$

and

$$k_i = m_i \qquad \text{for } i = 1, 2, \ldots, n - 1$$

with

$$k_i = 0 = m_i \quad \text{for } i = 2n - N, 2n - N + 1, \ldots, n$$

when

$$N \in \{n, n + 1, \ldots, 2n - 1\}.$$

### 3. SOLUTIONS WHEN $m = 2n$

When $m = 2n$, and $\{x_s\}$ is a solution set, the $x_{2i-1}$ ($i = 1, 2, \ldots, n$) and the $x_{2i}$ ($i = 1, 2, \ldots n$) of a solution set $\{x_s\}$ will be the roots of two polynomial equations of degree $n$ which we write as

$$R_n(X) + k_1 R_{n-1}(X) + \cdots + k_{n-1} R_1(X) + k_n = 0 \quad (14)$$

and

$$Q_n(X) + m_1 Q_{n-1}(X) + \cdots + m_{n-1} Q_1(X) + m_n = 0 \quad (15)$$

respectively. $\{R_n\}$ and $\{Q_n\}$ are two families of orthogonal polynomials with $\deg[R_n] = n = \deg[Q_n]$. These polynomials are described fully in the Appendix. Again, as the roots of these equations are a solution set, which satisfy (2), the constants $k_i$ and $m_j$ are real and are such that the roots of (14) and (15) lie in the interval $[-1, 1]$.

With variables

$$x_{2i-1} = \cos\theta_i \quad (i = 1, 2, \ldots, n)$$

and

$$x_{2i} = \cos\alpha_i \quad (i = 1, 2, \ldots, n),$$

Eqs. (1) may be written as

$$
\begin{aligned}
&\cos\theta_1 + \cos\theta_2 + \cdots + \cos\theta_n\\
&\quad = \cos\alpha_1 + \cos\alpha_2 + \cdots + \cos\alpha_n + 1\\
&\cos 2\theta_1 + \cos 2\theta_2 + \cdots + \cos 2\theta_n\\
&\quad = \cos 2\alpha_1 + \cos 2\alpha_2 + \cdots + \cos 2\alpha_n\\
&\qquad \vdots\\
&\cos N\theta_1 + \cos N\theta_2 + \cdots + \cos N\theta_n\\
&\quad = \cos N\alpha_1 + \cos N\alpha_2 + \cdots + \cos N\alpha_n\\
&\qquad + \frac{1}{2}[1 + (-1)^{N-1}].
\end{aligned} \quad (16)
$$

With the transformation $X = \cos\theta$, Eq. (14) may be written (see the Appendix) as

$$
\begin{aligned}
&\frac{\cos(n + 1/2)\theta}{\cos(\theta/2)} + k_1 \frac{\cos(n - 1/2)\theta}{\cos(\theta/2)} + \cdots\\
&\qquad + k_{n-1} \frac{\cos(3\theta/2)}{\cos(\theta/2)} + k_n = 0
\end{aligned} \quad (17)
$$

which has roots $\cos\theta_1, \cos\theta_2, \ldots, \cos\theta_n$, and with the transformation $X = \cos\alpha$, (15) may be written as

$$
\begin{aligned}
&\frac{\sin(n + 1/2)\alpha}{\sin\alpha/2} + m_1 \frac{\sin(n - 1/2)\alpha}{\sin\alpha/2} + \cdots\\
&\qquad + m_{n-1} \frac{\sin(3\alpha/2)}{\sin(\alpha/2)} + m_n = 0
\end{aligned} \quad (18)
$$

which has roots $\cos\alpha_1, \cos\alpha_2, \ldots, \cos\alpha_n$.

We now show that

$$k_i = m_i \qquad i = 1, 2, \ldots, N$$

when

$$N \leq n$$

and

$$k_i = m_i \qquad i = 1, 2, ..., n$$

with

$$k_i = 0 = m_i \quad \text{for } i = 2n - N + 1, 2n - N + 2, ..., n$$

when

$$N \in \{n + 1, n + 2, ..., 2n\}.$$

We consider the complex variable

$$z = \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}$$

so that

$$z^r + z^{-r} = 2\cos\frac{r\theta}{2} \qquad \text{for all } r \in \mathbb{N}.$$

Substituting for $\cos r\theta/2$ in (17) gives

$$\frac{1}{z^{2n}}\left[\frac{z^{4n+2}+1}{z^2+1} + k_1 z^2\frac{z^{4n-2}+1}{z^2+1} + \cdots + k_{n-1}z^{2n-2}\frac{z^6+1}{z^2+1} + k_n z^{2n}\right] = 0. \quad (19)$$

It can be seen that the equation

$$z^{4n+2} + k_1 z^{4n} + \cdots + k_n z^{2n+2} + k_n z^{2n} + \cdots + k_1 z^2 + 1 = 0 \quad (20)$$

has the same roots as (19), together with two additional roots at $z^2 = -1$. Writing $\omega = z^2$, (20) becomes

$$\omega^{2n+1} + k_1\omega^{2n} + \cdots + k_n\omega^{n+1} + k_n\omega^n + \cdots + k_1\omega + 1 = 0. \quad (21)$$

Hence, the $2n + 1$ roots of this equation are

$$\omega_{2i-1} = \cos\theta_i + i\sin\theta_i, \omega_{2i} = \cos\theta_i - i\sin\theta_i,$$
$$(i = 1, 2, ..., n)$$
$$\omega_{2n+1} = -1,$$

where the $\cos\theta_i$ are the $n$ roots of (17). It follows that the $\cos\theta_i$ are related to the roots $\omega_i$ of (21) by

$$2\sum_{i=1}^{n}\cos r\theta_i = \sum_{i=1}^{2n}\omega_i^r = \left[\sum_{i=1}^{2n+1}\omega_i^r\right] + 1 \qquad \text{when } r = 1, 3, 5, ....$$
$$2\sum_{i=1}^{n}\cos r\theta_i = \sum_{i=1}^{2n}\omega_i^r = \left[\sum_{i=1}^{2n+1}\omega_i^r\right] - 1 \qquad \text{when } r = 2, 4, 6, .... \quad (22)$$

Similarly, by considering the complex variables

$$z = \cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}$$

and

$$\omega = z^2$$

it can be shown that the $2n + 1$ roots of the equation

$$\omega^{2n+1} + m_1\omega^{2n} + \cdots + m_n\omega^{n+1} - m_n\omega^n - \cdots - m_1\omega - 1 = 0$$

are $\omega_{2i-1} = \cos\alpha_i + i\sin\alpha_i$, $\omega_{2i} = \cos\alpha_i - i\sin\alpha_i$, (23)

and $\omega_{2n+1} = 1 \quad (i = 1, 2, ..., n)$,

where the $\cos\alpha_i$ are the $n$ roots of (18), so that these roots and the $2n + 1$ roots of (23) are such that

$$2\sum_{i=1}^{n}\cos r\alpha_i = \sum_{i=1}^{2n}\omega_i^r = \left[\sum_{i=1}^{2n+1}\omega_i^r\right] - 1 \quad (24)$$

for all $r \in \mathbb{N}$.

The relationships (16), (22), and (24) show that the first $N$ coefficients of (21) and (23) must be equal.

Thus,

$$k_i = m_i \qquad \text{for} \qquad i = 1, 2, ..., N$$

when

$$N \leq n$$

and

$$k_i = m_i \qquad \text{for} \qquad i = 1, 2, ..., n$$

with

$$k_i = 0 = m_i \quad \text{for } i = 2n - N + 1, 2n - N + 2, ..., n$$

when

$$N \in \{n + 1, n + 2, ..., 2n\}.$$

## 4. SOLUTION SETS

We have shown that the roots of Eqs. (3) and (4), or (14) and (15), will satisfy Eqs. (1). In each case the polynomial equations contain an "unknown" $m - N$ constants.

By assigning values, $b_1, b_2, \ldots, b_{m-N}$ say, to an arbitrary $m - N$ of the $x_s$'s, $Y_1, Y_2, \ldots, Y_{m-N}$ say, and substituting these values into the appropriate equations (i.e. (3) or (14) if we wish to specify $x_{2i-1} = b_r$, (4) or (15) if we wish to specify $x_{2i} = b_s$), we obtain $m - N$ linear equations which may be solved to find the values of the constants $k_i$, $m_j$ in terms of the known values $b_j$. Thus, by examining the roots of these polynomial equations, it is possible to determine whether or not a solution set $\{x_s\}$, satisfying both Eqs. (1) *and* the inequalities (2) can be found in which $m - N$ of the $x_s$'s take their assigned values. For certain combinations of $Y_i$ and $b_j$, solutions satisfying (2) cannot be found, and no solution set exists.

We now consider what restrictions must be placed on the $b_i$'s to ensure that a solution set can be found for the case $N = m - 1$.

Suppose that $m = 2n - 1$ and $N = m - 1 = 2n - 2$ ($n \in \mathbf{N}$). Then Eqs. (3) and (4) are

$$T_n(X) + k_1 T_{n-1}(X) = 0 \tag{25}$$

and

$$U_{n-1}(X) + k_1 U_{n-2}(X) = 0, \tag{26}$$

where $k_1$ is an arbitrary constant whose value is found by assigning a value to one of the $X_s$'s. If $x_{2i-1}$ is to take the value $b$, say, then

$$k_1 = -\frac{T_n(b)}{T_{n-1}(b)},$$

and if $x_{2i}$ is to take the value $b$ then

$$k_1 = -\frac{U_{n-1}(b)}{U_{n-2}(b)}.$$

We now look at the behaviour of the roots of Eqs. (25) and (26) as $k_1$ varies. If $X = \cos\theta$, where $\theta \in [0, \pi]$, satisfies equation (25), it can be shown that

$$\frac{dk_1}{d\theta} = \frac{1}{2}\sec^2(n-1)\theta\,[(2n-1)\sin\theta + \sin(2n-1)\theta]$$

so that

$$\frac{dk_1}{dX} = -\frac{1}{2}\sec^2(n-1)\theta\left[(2n-1) + \frac{\sin(2n-1)\theta}{\sin\theta}\right]$$
$$= -\frac{1}{2}\frac{((2n-1) + U_{2n-2}(X))}{(T_{n-1}(X))^2}$$

It can be shown that

$$\frac{dk_1}{dX} < 0 \quad \text{for all } n \text{ and } X \in [-1, 1].$$

Hence, if $X = x_s$ is a solution of (25), then $x_s$ decreases monotonically with $k_1$. Similarly, it can be shown that if $X = x_s$ is a solution of (26), then $x_s$ decreases monotonically with $k_1$.

Thus, all roots $x_1, x_2, \ldots, x_{2n-1}$ decrease as $k_1$ increases. As we are seeking solutions which satisfy (2), we now consider the two limiting cases in which we specify either $x_1 = 1$ or $x_{2n-1} = -1$.

Suppose $x_1 = 1$ is a solution. Then $X = 1$ must be a root of (25), so that

$$T_n(1) + k_1 T_{n-1}(1) = 0$$

which implies $k_1 = -1$.

With $k_1 = -1$, the roots of (25) are given by

$$\begin{aligned} x_1 &= 1 \\ x_{2s+1} &= \cos\left(\frac{2s\pi}{2n-1}\right) \quad s = 1, 2, \ldots, n-1 \end{aligned} \tag{27}$$

and the roots of (26) are

$$x_{2s} = \cos\left[\frac{(2s-1)\pi}{2n-1}\right] \quad s = 1, 2, \ldots, n-1 \tag{28}$$

and it can be seen that these are a solution set.

Now suppose that $x_{2n-1} = -1$ is a solution. Then $X = -1$ will be a root of (25) and

$$T_n(-1) + k_1 T_{n-1}(-1) = 0$$

which implies $k_1 = 1$.

With $k_1 = 1$, the roots of (25) are given by

$$x_{2s-1} = \cos\left[\frac{(2s-1)\pi}{2n-1}\right] \quad s = 1, 2, \ldots, n \tag{29}$$

and the roots of (26) are

$$x_{2s} = \cos\left[\frac{2s\pi}{2n-1}\right] \quad s = 1, 2, \ldots, n-1 \tag{30}$$

and these again are a solution set.

Hence, as all roots decrease monotonically with $k_1$, it follows that $k_1 \in [-1, 1]$ for all roots to lie in the interval $[-1, 1]$, and furthermore it can be seen from (27)–(30), that for any $k_1$ lying in this interval,

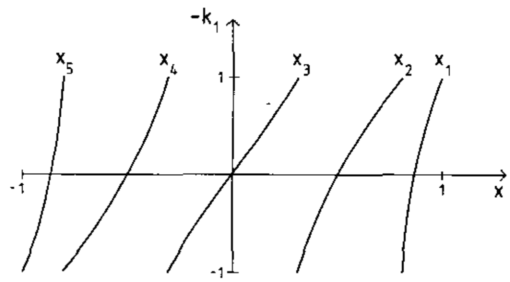


**FIG. 3.** $x_s$ as a function of $k_1$ for an $m = 5$, $N = 4$ gradiometer. Typically one would select a value of $x_1$ based on external response criteria; this determines $k_1$ and hence the positions of the other four crossovers such that gradiometric order is maintained.

$$\cos\left(\frac{s\pi}{m}\right) \le x_s \le \cos\left(\frac{(s-1)\pi}{m}\right) \quad (31)$$

for all $s = 1, 2, \ldots m$, and a solution set exists. Figure 3 illustrates the behaviour of the $x_s$ for $m = 5$ and $N = 4$.

Thus, if we specify $x_s = b$, a solution set will exist if and only if

$$\cos\left[\frac{s\pi}{m}\right] \le b \le \cos\left[\frac{(s-1)\pi}{m}\right]. \quad (32)$$

Similarly, when $m = 2n$, it can be shown that all roots $x_1, \ldots, x_{2n}$ decrease monotonically with $k_1$, solution sets exist if and only if $k_1 \in [-1, 1]$, and that (31) and (32) hold in this case too.

Subject to the existence of the solution set, we are thus free to choose any value of $b$ in the appropriate interval. Specifically, we can choose a value $b$ for $x_1$ resulting in a zero or a peak in the spatial sensitivity of the device at a given location. Indeed in the case $m = N + 2$ we can modify the response asymmetrically at the two ends of the gradiometer [7].

## CONCLUSIONS

We have discussed the problem above in terms of planar magnetic gradiometers, for near-source discrimination. Examples of the modifications to near-source sensitivity as described in the Introduction have been previously published [7]. However, we should emphasise that the method is quite general and may be applied to other situations. For example, the simplest form of superconducting gradiometer (i.e., $N = m$) is used in magnetic monopole detectors [10, 12], but a potential interference source is the wall of the enclosing liquid helium vessel; the above formalism allows one to minimise the response to this source while maintaining the necessary high gradiometric order. Again, although we have described the measurement of tangential fields, the analysis is also applicable to planar gradiometers used as radial field sensors (i.e. in terms of Fig. 1 a planar gradiometer whose plane is parallel to the $YZ$ plane). In this case the ability to discriminate against the SQUID is retained, although other modifications to the gradiometer response are not so obviously advantageous. We believe there may also be applications in feature recognition algorithms; although we have described the system as a "spatial filter," it could readily be applied to time-series analysis. We plan to investigate this approach further, particularly the effects of designs asymmetric around $x = 0$).

## APPENDIX: PROPERTIES OF THE POLYNOMIALS $R_n$ AND $Q_n$.

1. The polynomials $R_n$ are orthogonal on the interval $[-1, 1]$ with respect to the weight function

$$w_R(X) = \left[\frac{1+X}{1-X}\right]^{1/2}$$

with

$$\int_{-1}^{1} w_R(X)\, R_m(X)\, R_n(X)\, dX = \pi \quad \text{if } m = n$$
$$0 \quad \text{if } m \ne n.$$

The polynomials $Q_n$ are orthogonal on $[-1, 1]$ with respect to the weight function

$$w_Q(X) = \left[\frac{1-X}{1+X}\right]^{1/2}$$

with

$$\int_{-1}^{1} w_Q(X)\, Q_m(X)\, Q_n(X)\, dX = \pi \quad \text{if } m = n$$
$$0 \quad \text{if } m \ne n.$$

2. $R_n$ and $Q_n$ satisfy the recurrence relationships

$$R_{n+1}(X) = 2X\, R_n(X) - R_{n-1}(X) \quad n = 1, 2, 3\ldots$$
$$Q_{n+1}(X) = 2X\, Q_n(X) - Q_{n-1}(X) \quad n = 1, 2, 3\ldots$$

with

$$R_o(X) = 1, \quad R_1(X) = 2X - 1,$$
$$Q_o(X) = 1, \quad Q_1(X) = 2X + 1.$$

3. $R_n$ and $Q_n$ satisfy the differential equations

$$(1 - X^2)\, R_n''(X) + (1 - 2X)\, R_n'(X) + n(n+1)\, R_n(X) = 0$$
$$(1 - X^2)\, Q_n''(X) - (1 - 2X)\, Q_n'(X) + n(n+1)\, Q_n(X) = 0.$$

4. $R_n$ and $Q_n$ are given explicitly by

$$R_n(X) = \sum_{m=0}^{n} (-1)^{n-m} 2^m \frac{2n+1}{2m+1} \binom{n+m}{n-m} (1+X)^m$$

$$Q_n(X) = \sum_{m=0}^{n} (-1)^{n-m} 2^m \binom{n+m}{n-m} (1+X)^m.$$

5. $R_n$ and $Q_n$ are related to the Chebyshev polynomials by

$$X R_n(2X^2 - 1) = T_{2n+1}(X)$$

$$Q_n(2X^2 - 1) = U_{2n}(X)$$

and it can be seen that

$$R_n(\cos\theta) = \frac{T_{2n+1}(\cos(\theta/2))}{\cos(\theta/2)} = \frac{\cos(n+\frac{1}{2})\theta}{\cos(\theta/2)}$$

$$Q_n(\cos\theta) = U_{2n}\left[\cos\frac{\theta}{2}\right] = \frac{\sin(n+\frac{1}{2})\theta}{\sin(\theta/2)}.$$

6. $R_n$ and $Q_n$ are related to the Jacobi polynomials

$$P_n^{(-1/2,1/2)}(X) \quad \text{and} \quad P_n^{(1/2,-1/2)}(X)$$

by

$$\binom{2n}{n} R_n(X) = 2^{2n} \quad P_n^{(-1/2,1/2)}(X)$$

$$\binom{2n}{n} Q_n(X) = 2^{2n} \quad P_n^{(1/2,-1/2)}(X).$$

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## REFERENCES

1. S. Evanson, R. J. P. Bain, G. B. Donaldson, G. Stirling, and G. Hayward, *IEEE Trans. Magn.* **MAG25,** 1200 (1989).
2. A. Cochran, G. B. Donaldson, S. Evanson, and R. J. P. Bain, *Proc. IEEE A* **140,** 113 (1993)
3. P. Carelli and V. Pizzella, *Supercond. Sci. Tech.* **5**, 407 (1992).
4. G. L. Romani and R. Leoni, *Biomagnetism: Application and Theory*, edited by H. Weinberg, G. Stroinh, and T. Katila (Pergamon Press, New York, 1985).
5. R. Ilmoniemi, J. Knuutila, T. Ryhanen, and H. Seppa, *Prog. Low Temp. Phys.* **12,** 271 (1989).
6. Y. Uchikawa, F. Matsumura, K. Kobayashi, and M. Kotani, Application to Discrimination of Multisources to Somatosensory Evoked Field Using 3-dimensional Second-order Gradiometer, to be published in *Proc. 9th Int. Conf. on Biomagnetism, Vienna 1993* (Elsevier, Amsterdam, 1994).
7. R. J. P. Bain, A. E. Jones, and G. B. Donaldson, *IEEE Trans. Magn.* **MAG-10,** 100 (1987).
8. G. B. Donaldson, C. M. Pegrum, and R. J. P. Bain, Integrated Thin Film SQUID Instruments, in *SQUID'85—Superconducting Quantum Interference Devices and their Applications*, edited by H. D. Hahlbohm and H. Lubbig (W. de Gruyter, Berlin, 1985).
9. M. Abramowitz and J. A. Stegum, *Handbook of Mathematical Functions* (Dover Publications, New York, 1965), Chapter 22.
10. G. B. Donaldson and R. J. P. Bain, *Appl. Phys. Lett.* **45**, 990 (1984).
11. C. V. Durell and A. Robson, *Advanced Algebra* Vol.2, (G. Bell and Sons, London, 1958) p. 302.
12. C. D. Tesche, The IBM Monopole Experiments, in *Proceedings, 4th Workshop on Grand Unification*, edited by H. A. Weldon *et al.* (Birkhauser, New York, 1983).